Lecture 32

<u>Ex</u>: Compute the integral of $f(x,y) = 18y^3$ along the piece of the curve $x = y^3$ from $(-1,-1)$ to $(1,1)$.

<u>Sol</u>: We begin by parametrizing the curve:

letting $y = t \Rightarrow x = t^3$, so $\vec{r}(t) = \langle t^3, t \rangle$.

$\vec{r}(-1) = \langle -1,-1 \rangle$ and $\vec{r}(1) = \langle 1,1 \rangle$, so $-1 \le t \le 1$.

Now, $\vec{r}\,'(t) = \langle 3t^2, 1 \rangle$, so $|\vec{r}\,'(t)| = \sqrt{9t^4+1}$.

Thus, $\displaystyle\int_C f(x,y)\,ds = \int_{-1}^{1} f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$

$$= \int_{-1}^{1} 18(t)^3\left(\sqrt{9t^4+1}\right)dt = \int_{-1}^{1} 18t^3\sqrt{9t^4+1}\,dt$$

$$\begin{pmatrix} u = 9t^4+1 \\ du = 36t^3\,dt \end{pmatrix} \quad \int_{10}^{10} \sqrt{u}\left(\tfrac{1}{2}du\right) = 0.$$

$\diamond$

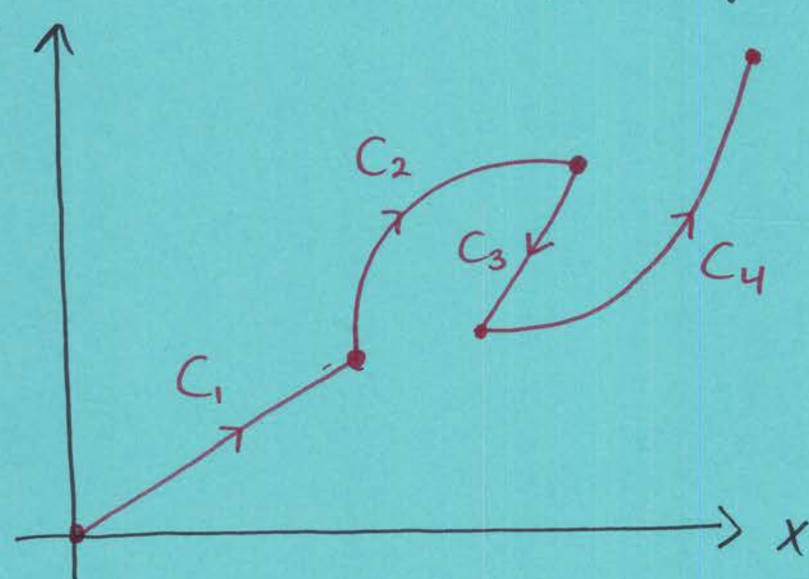
Line integrals in 3 variables have a similar formula. For a smooth curve C in $\mathbb{R}^3$ parametrized by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \le t \le b$, the line integral is given by:

$$\boxed{\int_C f(x,y,z)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt = \int_a^b f(x(t),y(t),z(t))\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\,dt}$$

Integrals of the form $\int_C f\,ds$ are called <u>line integrals with respect to arclength</u> (s).

(I, personally, like to call these <u>scalar line integrals</u>.)

If C is a <u>piecewise-smooth curve</u>, i.e., a collection of smooth curves $C_1, \cdots, C_n$ joined end on end, e.g.,



then, we have $$\int_C f\,ds = \int_{C_1} f\,ds + \cdots + \int_{C_n} f\,ds.$$

<u>An application of scalar line integrals</u>:

Suppose we have a thin wire bent in the shape of a curve C in $\mathbb{R}^2$ (or $\mathbb{R}^3$), and that the linear density of the wire is represented by the function $\rho(x,y)$ (or $\rho(x,y,z)$). Then the <u>mass</u> of the wire is given by $\boxed{m = \int_C \rho\,ds}$

And the center of mass of the wire has coordinates: $\boxed{\bar{x} = \frac{1}{m}\int_C x\rho\,ds\ ,\ \bar{y} = \frac{1}{m}\int_C y\rho\,ds}$

(and, if in $\mathbb{R}^3$: $\boxed{\bar{z} = \frac{1}{m}\int_C z\rho\,ds}$).

Ex: Find the mass of the wire bent in the shape of the helix $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$, $0 \le t \le 5$, given that the linear density of the wire is $\rho(x,y,z) = x^2 + y^2 + (z+1)^2$.

Sol: The mass is: $m = \int_C \rho\,ds = \int_0^5 \rho(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$.

$$\rho(\vec{r}(t)) = (\cos t)^2 + (\sin t)^2 + (t+1)^2 = 1 + (t+1)^2$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1\rangle \Rightarrow |\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

So, $m = \int_0^5 (1 + (t+1)^2)(\sqrt{2})\,dt = \sqrt{2}\left(t + \frac{1}{3}(t+1)^3\right)\Big|_0^5$

$$= \sqrt{2}\left[\left(5 + \tfrac{1}{3}(6)^3\right) - \left(0 + \tfrac{1}{3}(1)^3\right)\right] = \sqrt{2}\left(77 - \tfrac{1}{3}\right) = \frac{\sqrt{2}(230)}{3}$$

$\diamond$

In the definition of $\int_C f\,ds$, if we replace Δs_i by Δx_i or Δy_i, we can get two other types of line integrals:

- The line integral of f along C with respect to x:

$$\boxed{\int_C f(x,y)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta x_i}$$

- The line integral of f along C with respect to y:

$$\boxed{\int_C f(x,y)\,dy = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta y_i}$$

If C is parametrized by $\vec{r}(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$, then $dx = d(x(t)) = x'(t)dt$, $dy = d(y(t)) = y'(t)dt$

so: $$\boxed{\int_C f(x,y)\,dx = \int_a^b f(\vec{r}(t))\,x'(t)\,dt = \int_a^b f(x(t), y(t))\,x'(t)\,dt}$$

& $$\boxed{\int_C f(x,y)\,dy = \int_a^b f(\vec{r}(t))\,y'(t)\,dt = \int_a^b f(x(t), y(t))\,y'(t)\,dt}$$

We can even consider the case where these two happen together by using:

$$\boxed{\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_C P(x,y)\,dx + \int_C Q(x,y)\,dy}$$

As before, all this makes sense in the three variable case, where: $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \le t \le b$, &

$$\boxed{\int_C f(x,y,z)\,dz = \int_a^b f(x(t),y(t),z(t))\,z'(t)\,dt}$$

and $\int_C f(x,y,z)\,dx$ & $\int_C f(x,y,z)\,dy$ are defined likewise.

<u>Ex</u>: Compute the line integral $\int_C y\,dx - x\,dy + z\,dz$ where C is the path given by:

ⓐ C_1: $\vec{r}_1(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 2\pi$.

ⓑ C_2: line segment from $(1,0,0)$ to $(1,0,2\pi)$.

ⓒ C_3: line segment from $(1,0,2\pi)$ to $(1,0,0)$.

<u>Sol</u>: ⓐ Simply plug $\vec{r}_1(t)$ in:

$$\int_{C_1} y\,dx - x\,dy + z\,dz = \int_0^{2\pi} \Big((\sin t)\,d(\cos t) - (\cos t)\,d(\sin t) + (t)\,d(t)\Big)$$

$$= \int_0^{2\pi} \Big((\sin t)(-\sin t\,dt) - (\cos t)(\cos t\,dt) + t\,dt\Big) = \int_0^{2\pi} (-1 + t)\,dt$$

$$= \left(-t + \tfrac{1}{2}t^2\right)\Big|_0^{2\pi} = -2\pi + 2\pi^2 = 2\pi(\pi - 1)$$

(b) First parametrize the line segment:

[Recall an easy way to parametrize a line segment is $\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1,\ 0 \le t \le 1$ where $\vec{r}_0$ is the position vector of the initial point & $\vec{r}_1$ is the position vector of the ending point.]

For C_2: $\vec{r}_2(t) = \langle 1, 0, 2\pi t \rangle,\ 0 \le t \le 1$. So:

$$\int_{C_2} y\,dx - x\,dy + z\,dz = \int_0^1 \left[(0)\,d(1) - (1)\,d(0) + (2\pi t)\,d(2\pi t)\right]$$

$$= \int_0^1 \left[0 - 0 + 4\pi^2 t\right] dt = \int_0^1 4\pi^2 t\,dt = 2\pi^2 t^2 \Big|_0^1 = 2\pi^2$$

(c) C_3: $\vec{r}_3(t) = \langle 1, 0, 2\pi(1-t) \rangle,\ 0 \le t \le 1$. So:

$$\int_{C_3} y\,dx - x\,dy + z\,dz = \int_0^1 \left[(0)\,d(1) - (1)\,d(0) + (2\pi(1-t))\,d(2\pi(1-t))\right]$$

$$= \int_0^1 \left[0 - 0 + (2\pi(1-t))(-2\pi)\right] dt = \int_0^1 -4\pi^2(1-t)\,dt$$

$$= 2\pi^2(1-t)^2 \Big|_0^1 = 0 - 2\pi^2 = -2\pi^2$$

$\diamond$

So, what is the moral of this story?

In parts (a) & (b), we computed the line integral along different paths from $(1,0,0)$ to $(1,0,2\pi)$ and got different answers. This means that these types of line integrals (with respect to x, y, z) depend on the path of integration!

Also, notice that the line segments in (b) & (c) are the same ones, but they are traversed in opposite directions. To quantify this, we say that C_2 and C_3 have opposite orientations. (An orientation on a curve C is a choice of direction to traverse it. Orientations can be picked up by parametrizations.) We write $C_3 = -C_2$. Integrating C_3 gave us the negative of the integral along C_2.

In general, we have

- $\int_{-C} f\,dx = -\int_C f\,dx$
- $\int_{-C} f\,dy = -\int_C f\,dy$
- $\int_{-C} f\,dz = -\int_C f\,dz$

However, $\int_{-C} f\,ds = \int_C f\,ds$ since arc length only depends on the curve and not the direction traveled.